# M.A./M.Sc. Examination, 2017
# Semester-IV
# Mathematics
# Course: MMO-41
# ( Advanced Real Analysis-II )

**Time: Three Hours** **Full Marks: 40**

*Questions are of value as indicated in the margin.*

Answer ***Question No. 6*** and ***any three*** from the rest.

1. a) Define $\sigma$- algebra? Give an example of a nontrivial $\sigma$- algebra. 3

   b) Let $(X, \mathcal{A}, \mu)$ be a measure space. If $A, B \in \mathcal{A}$, with $B \subset A$, show that

   i) $\mu(B) \leq \mu(A)$

   ii) if $\mu(B) < \infty$ then $\mu(A - B) = \mu(A) - \mu(B)$ 2

   c) When is a sequence of sets called convergent? Prove that a monotone increasing sequence of sets is convergent and converges to its supremum. 5

   d) Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where $\mu$ is counting measure. Let $A_n = \{n, n+1, n+2, ...\}$. Show that $\lim_{n\to\infty} \mu(A_n) \neq \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$. 2

2. a) Let $\mu^*$ be an outer measure on a set $X$. Let $\mathcal{M}$ be the class of all $\mu^*$- measurable sets. Show that $\mathcal{M}$ is an $\sigma$- algebra and $\mu^*$- restricted to $\mathcal{M}$ is a measure. 6

   b) When is an outer measure $\mu^*$ on $X$ said to be regular? Let $\mu^*$ be a regular outer measure on $X$ and let $\mu^*(X) < \infty$. Prove that a necessary and sufficient condition that a set $A \subset X$ be measurable is $\mu^*(X) = \mu^*(A) + \mu^*(X - A)$. 6

3. a) Show that there is an outer measure on $\mathbb{R}^2$, constructed from premeasure such that under which open sets in $\mathbb{R}^2$ need not be measurable. 4

   b) When is a measure $(X, \mathcal{M}, \mu)$ said to be complete? Show that there is a complete measure space $(X, \bar{\mathcal{M}}, \bar{\mu})$ such that $\mathcal{M} \subset \bar{\mathcal{M}}$ and $\bar{\mu}$ is extension of $\mu$ on $\bar{\mathcal{M}}$. 6

   c) Show that the Lebesgue measure is complete. 2

4. a) What is simple function. If $f$ is nonnegative measurable function then show that there is a sequence of nondecreasing simple function which converges to $f$. 1+7

   b) Show that sum and product of two simple functions are simple. 4

5. a) If $f$ is integrable on $E$ and $F \subset E$, then show that $f$ is integrable on $F$. Moreover if $f$ is nonnegative on $E$ then show that $\int_F f \leq \int_E f$. 5

   b) If $f$ is integrable on $E$ and $c$ is constant, show that $cf$ is integrable on $E$ and $\int_E cf = c\int_E f$. 5

   c) If $f$ is integrable on $E$ and $f = g$ a.e on $E$ then show that $\int_E f = \int_E g$. 2

**P.T.O.**

6. Answer ***any two***: 2x2

a) If $f$ be an integrable function defined on a measure space $(X, \mathcal{M}, \mu)$ such that $\int_E f\, d\mu = 0$ for every $E \in \mathcal{M}$, then show that $f = 0$ a.e in $X$.

b) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $E \in \mathcal{M}$. If $\{f_n\}$ and $\{g_n\}$ be sequences of finite a.e, measurable functions on $E$ such that $f_n \to f[\text{meas}]$ and $g_n \to g[\text{meas}]$ on $E$, show that $f_n + g_n \to f + g[\text{meas}]$ on $E$.

c) If $\mu(\text{E}) = 0$ then show that for any measurable function $f$, $\int_E f = 0$.

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