M.A./M.Sc. Examination, 2017
Semester-IV
Mathematics
Course: MMO-41
(Advanced Functional Analysis-II)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.

Answer any ***four*** questions.

1. a) Prove that every closed convex subset of a Hilbert space H has a unique member of smallest norm. 4
   b) Show that a Hilbert space H contains a complete orthonormal sequence iff H is separable. 4
   c) Let $M \subset H$ where H is a Hilbert space. Prove that $M^{\perp}$ (orthogonal complement of M) is a closed subspace of H. 2

2. a) Let f be a sesquilinear form defined on a Hilbert space and $\hat{f}$ be the associated quadratic form of f. Prove that f is bounded iff $\hat{f}$ is bounded and moreover $\|\hat{f}\| \le \|f\| \le 2\|\hat{f}\|$. 3
   b) Prove that for every bounded linear operator $T : H \to H$ where H is a Hilbert space its adjoint operator T^* exists satisfying $\|T\| = \|T^*\|$. 4
   c) Show that the collection of all self-adjoint operators defined on a real Hilbert space H form a closed subspace of $B(H, H)$ where $B(H, H)$ is the set of all bounded linear operators defined on H. 3

3. a) If N_1 and N_2 are normal operators defined on a Hilbert space H such that either commutes with the adjoint of the other. Prove that $N_1 + N_2$ and $N_1 N_2$ are normal operators on H. 4
   b) Let $T : H \to H$ be a self-adjoint operator where H is a Hilbert space. Show that $\|T\| = \sup\left\{\left|\langle Tx, x\rangle\right| : \|x\| = 1\right\}$. 4
   c) Prove that every self-adjoint operator is normal but converse may not be true. 2

4. a) Show that the spectrum $\sigma(T)$ of a bounded linear operator T on a complex Banach space X is compact (assume $\sigma(T) \ne \phi$). 5
   b) Prove that residual spectrum $\sigma_r(T)$ of a bounded self-adjoint linear operator $T : H \to H$ (where H is a complex Hilbert space) is empty. 3
   c) Let $T \in B(H, H)$ where H is a Hilbert space. Then prove that $\sigma(T^*) = \left\{k \in C : \bar{k} \in \sigma(T)\right\}$, symbols bear their usual meaning. 2

5. a) Let A be a complex Banach algebra with identity. Show that for any $x \in A$, the spectrum $\sigma(x) \ne \varphi$. 6
   b) Define a total orthonormal set in a Hilbert space. Establish the Bessel's inequality for an orthonormal set of vectors in a Hilbert space. 1+3

P.T.O.

6. a) Prove that spectral radius $r_\sigma(T)$ of a bounded linear operator T defined on a complex Banach space can be expressed as

$$r_\sigma(T) = \overline{\lim_{n\to\infty}} \sqrt[n]{\|T^n\|}.$$ 5

b) Let P_1 and P_2 be two projection operators on a Hilbert space H. Prove that $P_1 + P_2$ is a projection operator on H iff $P_1(H)$ and $P_2(H)$ are orthogonal. 3

c) Show that every unitary operator on a Hilbert space is isometric but not conversely. 2